Lecture 7

Plan of Lecture 7:

- Review of Lecture 6, §4.2 and point out a small typo in Lecture 6
- §4.3

Review of §4.2:

In Lecture 6 video (Lecture notes on course website fixed already),

sometimes, I misrote the eqn as

$$\text{“} ay'' + by' + c = 0 \text{”}$$

sometimes, I misrote the eqn as

$$\text{“} ay'' + by + c = 0 \text{”}, \text{ etc.}$$

They all should be

"$ay'' + by' + cy = 0$" !

In the last E.g in Lecture 6 video:

I wrote

"E.g: Recall $y'' + 4y' + y = 0$

The characteristic eqn

$$\lambda^2 + 4\lambda + \lambda = 0"$$

It really should be

"E.g Recall $y'' + 4y' + 4y = 0$

The characteristic eqn

$$\lambda^2 + 4\lambda + 4 = 0"$$

Let's go through this example again.

E.g: Consider

$$y'' + 4y' + 4y = 0 \qquad (*)$$

Its characteristic eqn is

$$\underbrace{\lambda^2 + 4\lambda + 4}_{= (\lambda+2)^2} = 0 \qquad (**)$$

$(**)$ has only one repeated root.

$$\lambda = -2.$$

In this case, we have two "L.I" solns of (1):

$$\begin{cases} y_1(x) = e^{-2x} \\ y_2(x) = x e^{-2x} \end{cases}$$

Then the general soln of $(*)$ is

$$y(x) = c_1 e^{-2x} + c_2 x e^{-2x}$$

For the purpose of HW/Exam, all you need to know about § 4.2 is:

Thm: Consider the D.E

$$ay'' + by' + cy = 0, \quad (1)$$

where $a, b, c \in \mathbb{R}$ and $a \neq 0$.

The characteristic eqn of (1) is

$$a\lambda^2 + b\lambda + c = 0 \quad (2)$$

Case (I): If (2) has two "distinct" real roots λ_1, λ_2

$\Delta > 0$

then the general soln of (1) is given by $y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$, $C_1, C_2 \in \mathbb{R}$

Case (II): If (2) has only one repeated real root λ_0

$\Delta = 0$

then the general soln of (1) is given by

$$y(x) = C_1 e^{\lambda_0 x} + C_2 x e^{\lambda_0 x}, \quad C_1, C_2 \in \mathbb{R}$$

Note: "$\mathbb{R}$": the set of real numbers

Why I write it as "$\mathbb{R}$" not R ?

"Let $R \in \mathbb{R}$"

a number a set

Similarly, "$\mathbb{C}$": the set of complex numbers

E.g.:

① Solve $y'' - 3y' + 2y = 0$ (3)

A: Its characteristic eqn is

$$\lambda^2 - 3\lambda + 2 = 0 \quad (4)$$

How to get (4) from (3)?

Change kth derivative $y^{(k)} \to \lambda^k$ ← kth power of λ

In particular, $y'' = y^{(2)} \to \lambda^2$

$y' \to \lambda$

$y = y^{(0)} \to \lambda^0 = 1$

Note (4) is

$$\lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) = 0$$

$\Rightarrow$ we have two distinct roots:

$$\lambda_1 = 1, \quad \lambda_2 = 2$$

$\Rightarrow$ The general soln of (3):

$$y(x) = C_1 e^x + C_2 e^{2x}.$$

E.g Solve I.V.P

$$y'' + 2y' + y = 0 \qquad (5)$$

$$\text{“} y(0) = 1, \quad y'(0) = 2 \text{”}$$

A: Its characteristic eqn:

$$\lambda^2 + 2\lambda + 1 = 0$$

$$\Rightarrow \quad (\lambda+1)^2 = 0$$

Thus it has only one repeated real root:

$$\lambda = -1$$

The general soln of (5) is

$$y(x) = c_1 e^{-x} + c_2 x e^{-x}$$

plug in to the initial condition:

$$y(0) = 1 \Rightarrow c_1 \cancel{e^{-0}}^{1} + c_2 \cdot \cancel{0 e^{-0}}^{0} = 1$$

$x=0$
$y=1$

$$\Rightarrow c_1 = 1$$

Note $y'(x) = -c_1 e^{-x} + c_2 e^{-x} - c_2 x e^{-x}$

$$\text{"}y'(0) = 2\text{"} \Rightarrow -c_1 + c_2 = 2$$

$x=0$
$y'=2$

$$\Rightarrow -1 + c_2 = 2 \Rightarrow c_2 = 3$$

Hence the soln to I.V.P is

$$y(x) = e^{-x} + 3xe^{-x}$$

Q: what if the characteristic eqn

$$a\lambda^2 + b\lambda + c = 0$$

has no real roots?

$$(\text{i.e. } \Delta = b^2 - 4ac < 0)$$

We will answer this Q in today's lecture.

- Even when $a\lambda^2 + b\lambda + c = 0$ has no real roots, it always has a complex root.

To explain this, we first recall

$\mathbb{C}$: the set of complex number

Key thing to know:

$$i = \sqrt{-1}$$

(In particular, $i^2 = (\sqrt{-1})^2 = -1$)

E.g

$$\sqrt{-4} = \sqrt{4 \cdot (-1)} = \sqrt{4} \cdot \sqrt{-1} = 2i$$

$$\sqrt{-9} = \sqrt{9}\,\sqrt{-1} = 3i$$

Every complex number $c \in \mathbb{C}$ can be written as

$c = a + bi$, (or $a + ib$)

where $a, b \in \mathbb{R}$.

E.g: $1+i$, $9+7i$, $\sqrt{2} + \sqrt{3}i$

E.g: $i(3+i) = i \cdot 3 + \cancel{i^2}^{-1} = -1 + 3i$

$(2+i)(3+i) = 2(3+i) + i(3+i)$

$= 6 + 2i + 3i + \cancel{i^2}^{-1}$

$= 5 + 5i$

☆ Important formula: "Euler's formula"

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad \theta \in \mathbb{R}$$

E.g: $e^{i\frac{\pi}{2}} = \cancel{\cos\frac{\pi}{2}}^{0} + i\cancel{\sin\frac{\pi}{2}}^{1} = i$

$e^{i\pi} = \cancel{\cos\pi}^{-1} + i\cancel{\sin\pi}^{0} = -1$

$$\Rightarrow \boxed{e^{i\pi} + 1 = 0}$$

" $0, 1, \pi, e, i$ "

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

E.g: $e^{1+\sqrt{3}i} = e^{1} e^{\sqrt{3}i}$ ← $\theta = \sqrt{3}$

$= e(\cos\sqrt{3} + i\sin\sqrt{3})$

$= e\cos\sqrt{3} + ie\sin\sqrt{3}$

Fact:

$e^{A+B} = e^{A} \cdot e^{B}$

for $A, B \in \mathbb{C}$

Now back to study D.E

E.g. Consider $y'' - y' + y = 0$. (6)

Its characteristic eqn

$$\lambda^2 - \lambda + 1 = 0$$

$a = 1$
$b = -1$
$c = 1$

Note we can always use the quadratic formula:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$\Delta = b^2 - 4ac$
$= 1 - 4 = -3$
< 0

$$= \frac{+1 \pm \sqrt{1^2 - 4}}{2}$$

$\sqrt{-3} = \sqrt{3} \cdot \sqrt{-1}$
$= \sqrt{3}\, i$

$$= \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm \sqrt{3}\, i}{2}$$

$$= \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$\alpha = \frac{1}{2}, \quad \beta = \frac{\sqrt{3}}{2}$

" $\alpha + \beta i$ $\alpha - \beta i$ "

Two complex roots: $\lambda_1 = \frac{1}{2} + \frac{\sqrt{3}}{2} i$, $\lambda_2 = \frac{1}{2} - \frac{\sqrt{3}}{2} i$

We can still use the previous idea:

$$y_1 = e^{\lambda_1 x} \quad ; \quad y_2 = e^{\lambda_2 x}$$

Note

- $y_1 = e^{\lambda_1 x} = e^{(\frac{1}{2} + \frac{\sqrt{3}}{2} i)x} = e^{\frac{x}{2}} \cdot e^{i \frac{\sqrt{3}}{2} x}$

$$= e^{\frac{x}{2}} \left(\cos\left(\frac{\sqrt{3}}{2} x\right) + i \sin\left(\frac{\sqrt{3}}{2} x\right) \right)$$

$$= e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2} x\right) + i e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2} x\right).$$

- $y_2 = e^{\lambda_2 x} = e^{(\frac{1}{2} - \frac{\sqrt{3}}{2} i)x} = e^{\frac{x}{2}} \cdot e^{i(-\frac{\sqrt{3}}{2} x)}$

$$= e^{\frac{x}{2}} \left(\cos\left(-\frac{\sqrt{3}}{2} x\right) + i \sin\left(-\frac{\sqrt{3}}{2} x\right) \right)$$

$$= e^{\frac{x}{2}} \left(\cos\left(\frac{\sqrt{3}}{2} x\right) - i \sin\left(\frac{\sqrt{3}}{2} x\right) \right)$$

$$\boxed{\cos\theta = \cos(-\theta)\,; \quad \sin(-\theta) = -\sin\theta}$$

But this is not a real valued function, and we want real valued function solutions. Is this possible?

Yes!

$c_1 y_1 + c_2 y_2$ is also a soln!

Note $\dfrac{y_1+y_2}{2} = \frac{1}{2}y_1 + \frac{1}{2}y_2$

is also a soln!

Compute

$$\frac{y_1+y_2}{2}$$

$$= \frac{\overbrace{\left(e^{\frac{x}{2}}\cos\frac{\sqrt{3}}{2}x + \cancel{ie^{\frac{x}{2}}\sin\frac{\sqrt{3}}{2}x}\right)}^{y_1} + \overbrace{\left(e^{\frac{x}{2}}\cos\frac{\sqrt{3}}{2}x - \cancel{ie^{\frac{x}{2}}\sin\frac{\sqrt{3}}{2}x}\right)}^{y_2}}{2}$$

$$= e^{x}\cos\left(\frac{\sqrt{3}}{2}x\right)$$

Similarly: $\frac{y_1 - y_2}{2i} = \frac{1}{2i} y_1 - \frac{1}{2i} y_2$

is also a soln!

Ex. Compute and verify

$$\frac{y_1 - y_2}{2i} = e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2} x\right)$$

Thus we get two new (real-valued!)

solns:

$$y_3 = e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2} x\right)$$

$$y_4 = e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2} x\right)$$

You can verify directly they are solns to (6): $y'' - y' + y = 0$

Ex: Verify $y_3(x) = e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}x\right)$

is a soln of

$$y'' - y' + y = 0 \quad (6)$$

A: $y_3'(x) = \frac{1}{2} e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}x\right) - \frac{\sqrt{3}}{2} e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2}x\right)$

$$y_3''(x) = \frac{1}{4} e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}x\right) - \frac{\sqrt{3}}{4} e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$- \frac{\sqrt{3}}{4} e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2}x\right) - \frac{3}{4} e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= -\frac{1}{2} e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}x\right) - \frac{\sqrt{3}}{2} e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2}x\right)$$

$\Rightarrow y_3'' - y_3' + y_3 = \cdots = 0$

Exercise!

Ex: Compute and verify that

$y_4 = e^{\frac{x}{2}} \sin(\frac{\sqrt{3}}{2}x)$ is a soln

of $y'' - y + y = 0$.

Ex: Verify wronskian

$W(y_3, y_4) = W\left(e^{\frac{x}{2}}\cos(\frac{\sqrt{3}}{2}x), e^{\frac{x}{2}}\sin(\frac{\sqrt{3}}{2}x)\right)$

$\underset{y_3y_4' - y_4y_3'}{\parallel}$

is NOT 0.

$\Rightarrow y_3 = e^{\frac{x}{2}}\cos(\frac{\sqrt{3}}{2}x), y_4 = e^{\frac{x}{2}}\sin(\frac{\sqrt{3}}{2}x)$ are L.I.

Now we have two L.I. Solns:

$$y_3 = e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}x\right), \quad y_4 = e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2}x\right)$$

$\Rightarrow$ The general solns of " $y'' - y' + y = 0$ "

is $y(x) = C_1 y_3 + C_2 y_4$

$$= C_1 e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}x\right) + C_2 e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2}x\right)$$

Here $C_1, C_2 \in \mathbb{R}$

In general, we have the following theorem:

Thm: Consider the D.E

$$ay'' + by' + cy = 0. \quad (1)$$

Its characteristic eqn is

$$a\lambda^2 + b\lambda + c = 0. \quad (2)$$

"$\Delta < 0$"

If (2) has two complex roots:

$$\lambda_1 = \alpha + \beta i, \quad \lambda_2 = \alpha - \beta i$$

then we have two L.I. solns to (1):

$$y_1 = e^{\alpha x}\cos(\beta x), \quad y_2 = e^{\alpha x}\sin(\beta x)$$

the general solns to (1) are:

$$y(x) = C_1 e^{\alpha x}\cos\beta x + C_2 e^{\alpha x}\sin\beta x.$$

E.g Solve the I.V.P

$$y'' - 2y' + 2y = 0 \quad (7)$$

"$y(0) = 1, \quad y'(0) = 1.$"

A: The characteristic eqn of (7) is

$$\lambda^2 - 2\lambda + 2 = 0$$

$a = 1$

$b = -2$

$c = 2$

The roots:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4 - 4 \cdot 2}}{2}$$

$$= \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2\sqrt{-1}}{2}$$

$= 1 \pm i$ $\quad \lambda_1 = 1+i, \quad \lambda_2 = 1-i$

$\qquad \alpha + \beta i \qquad \alpha - \beta i$

$\Rightarrow \quad \alpha = 1 \; ; \quad \beta = 1$

By the thm, the general soln:

$$y(x) = C_1 e^{\alpha x}\cos(\beta x) + C_2 e^{\alpha x}\sin(\beta x)$$

$$= C_1 e^{x}\cos x + C_2 e^{x}\sin x$$

$y(0) = 1 \Rightarrow C_1 \cdot \cancel{e^{0}}^{1} \cancel{\cos 0}^{1} + \underbrace{C_2 e^{0} \cancel{\sin 0}^{0}}$

$x = 0$

$y = 1$

$$= C_1 = 1$$

$$\Rightarrow C_1 = 1$$

Note

$$y'(x) = C_1 e^{x}\cos x - C_1 e^{x}\sin x$$

$$+ C_2 e^{x}\sin x + C_2 e^{x}\cos x$$

$x = 0$
$y' = 1$

$y'(0) = 1 \Rightarrow$

$$C_1 + C_2 = 1 \Rightarrow C_2 = 0$$

Hence the soln to I.V.P is

$$y = e^x \cos x$$